Solution exercice 5_1

(a) The layer is thin so vertical water movement can be ignored.

$$q_i = -K_i \frac{dH}{dx} = -K_i \frac{d\left(z+h\right)}{dx} = -K_i \frac{d\left(z-\psi\right)}{dx} = K_i \frac{d\psi}{dx}, i = 1, 2$$

(b) *For layer 1:*

$$q_1 = K_1(\psi) \frac{d\psi}{dx}$$

$$\therefore \int_0^{L/2} q_1 \, dx = \int_{\psi_0}^{\psi(L/2)} K_1(\psi) \, d\psi$$

$$\therefore -q_1 L/2 = \frac{K_{1s}}{\alpha_1} \left\{ \exp\left[-\alpha_1 \psi(L/2)\right] - \exp\left[-\alpha_1 \psi_0\right] \right\}$$

Similarly, for layer 2:

$$-q_2 L/2 = \frac{K_{2s}}{\alpha_2} \left\{ \exp\left(-\alpha_2 \psi_L\right) - \exp\left[-\alpha_2 \psi(L/2)\right] \right\}$$

Since the flow is steady, $q_1 = q_2$:

$$\frac{K_{2s}}{\alpha_2} \left\{ \exp\left(-\alpha_2 \psi_L\right) - \exp\left[-\alpha_2 \psi(L/2)\right] \right\} = \frac{K_{1s}}{\alpha_1} \left\{ \exp\left[-\alpha_1 \psi(L/2)\right] - \exp\left[-\alpha_1 \psi_0\right] \right\}$$

(c) The solution is found from iteration, $\psi\left(\frac{L}{2}\right) = 1.4$ m. More exactly, the result is $\psi(L/2) =$ 1.41155 m.

(d) Flow is from high head to low head. In this case z is constant so the flow is from low suction to high suction (elevation head is the same everywhere), i.e., right to left.

(e) Flux is given by q_1 or q_2 in part (b). The answer is $q = 3.44 \times 10^{-3}$ m/y.

Solution exercice 5_2

(a) The transformation is $\phi = xt^{-1/2}$. We write:

$$\theta(x,t) = \theta(\phi),$$

so

$$\frac{\partial\theta}{\partial x} = \frac{\partial\phi}{\partial x}\frac{d\theta}{d\phi} = \frac{1}{\sqrt{t}}\frac{d\theta}{d\phi} \text{ and } \frac{\partial}{\partial x} = \frac{1}{\sqrt{t}}\frac{d}{d\phi}$$

as well as

$$\frac{\partial\theta}{\partial t} = \frac{\partial\phi}{\partial t}\frac{d\theta}{d\phi} = -\frac{1}{2}\frac{x}{t\sqrt{t}}\frac{d\theta}{d\phi} = -\frac{\phi}{2}\frac{1}{t}\frac{d\theta}{d\phi}$$

Now substitute into the governing equation, $\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial\theta}{\partial x}\right]$:

$$-\frac{\phi}{2}\frac{1}{t}\frac{d\theta}{d\phi} = \frac{1}{\sqrt{t}}\frac{d}{d\phi}\left[D(\theta)\frac{1}{\sqrt{t}}\frac{d\theta}{d\phi}\right]$$

and so

$$-\frac{\phi}{2}\frac{d\theta}{d\phi} = \frac{d}{d\phi}\left(D(\theta)\frac{d\theta}{d\phi}\right).$$

The boundary conditions follow directly from the given boundary and initial conditions.

(b) The Bruce and Klute equation follows by converting from $\theta(\phi)$ to $\phi(\theta)$ in the above equation. Thus, multiply each side by $d\phi/d\theta$ to get

$$-\frac{\phi}{2} = \frac{d}{d\theta}\left(D(\theta)\frac{1}{d\phi/d\theta}\right).$$

Now integrate with respect to θ from $\theta = \theta_i$:

$$D(\theta) = -\frac{1}{2}\frac{d\phi}{d\theta}\int_{\theta_i}^{\theta}\phi(\overline{\theta})\, d\overline{\theta},$$

which is just the Bruce and Klute equation.

(c) If gravity plays no role (e.g., if we did an experiment in outer space), then all the moisture profiles would collapse to a single curve when plotted as $\theta(\phi)$ versus ϕ.

Solution exercice 5_3

On choisira l'axe des z orienté positivement vers le bas (origine = surface du sol), si bien que H = h – z

1ère mesure H_{80} = - 100 cm
H_{120} = - 170 cm ⇒ h_{moy1} = -35cm
$\Delta H = H_{80} - H_{120}$ = 70 cm $\Delta z = z_{80} - z_{120}$ = -40 cm $\Delta H/\Delta z$ = - 1.75

$K(h_{moy1})$ = 0.6 cm/j ⇒ Flux q = 1.05 cm/j > 0 ⇒ percolation

2ème mesure H_{80} = -140 cm
H_{120} -120 cm ⇒ h_{moy2} = -30cm
$\Delta H = H_{80} - H_{120}$ = -20 cm $\Delta z = z_{80} - z_{120}$ = -40 cm $\Delta H/\Delta z$ = 0.5

$K(h_{moy1})$ = 1.1 cm/j ⇒ Flux q = -0.55 cm/j < 0 ⇒ remontée capillaire

More precise answer for ex. 5_3

Darcy's law for steady flow is (z positive upwards):

$$q = -K(h)\frac{dH}{dz} = -K(h)\left(1 + \frac{dh}{dz}\right).$$

This is a first-order separable ODE for h(z). The solution is:

$$-\int_{z_A}^{z_B} d\bar{z} = z_A - z_B = \int_{h(z_A)}^{h(z_B)} \frac{d\bar{h}}{1 + \frac{q}{K(\bar{h})}}.$$

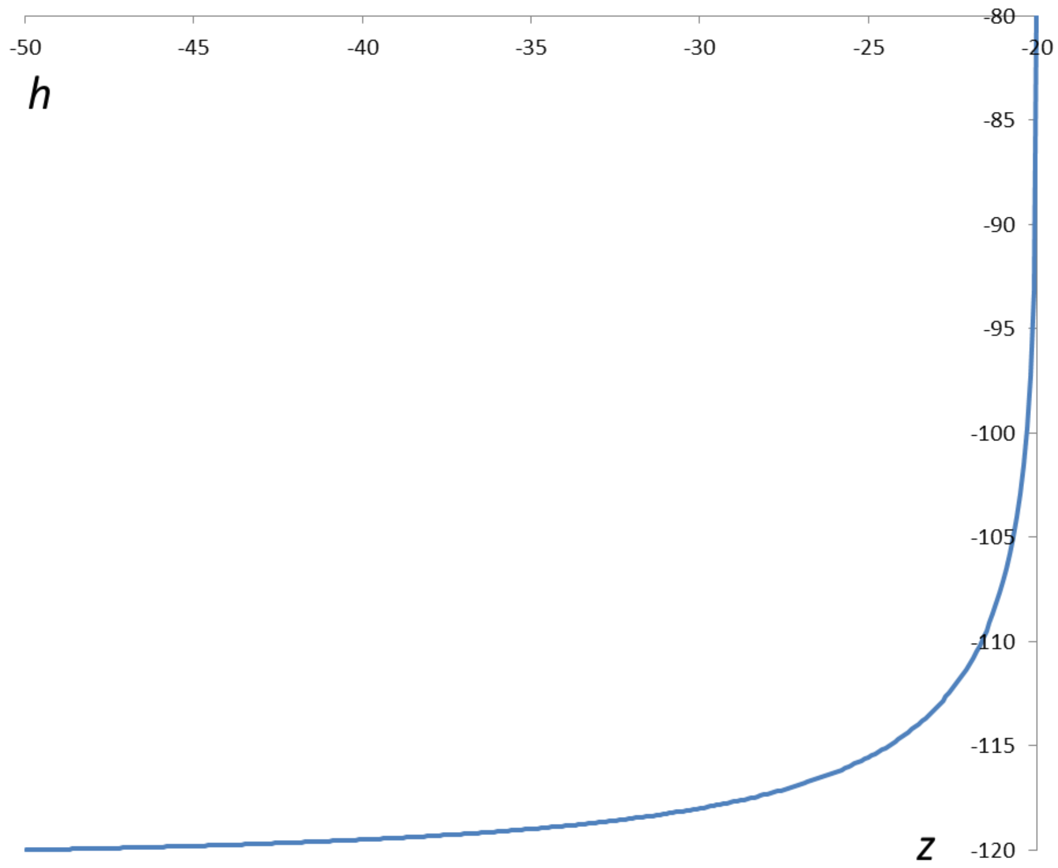
For the given question, take the soil surface as z = 0, then z_A = -80, z_B = -120, $h(z_A)$ = -20 and $h(z_B)$ = -50. The integral on the right cannot be evaluated easily. However, numerically, it is easily shown that q = -5.0119 cm/day (sign of the Darcy flux differs from above because there z is taken as positive downwards), i.e., flow is downwards (percolation). This is 5 times as large as the approximate answer given above. The reason is that K varies rapidly, so the average in the question is not appropriate. The exact answer is obtained for $h_{average}$ = -23.57 cm, not -35 cm as used in the solution above. There is no simple way to estimate this average a priori.

With q known, the integral above can be used to calculate h(z) also. Rewrite it as:

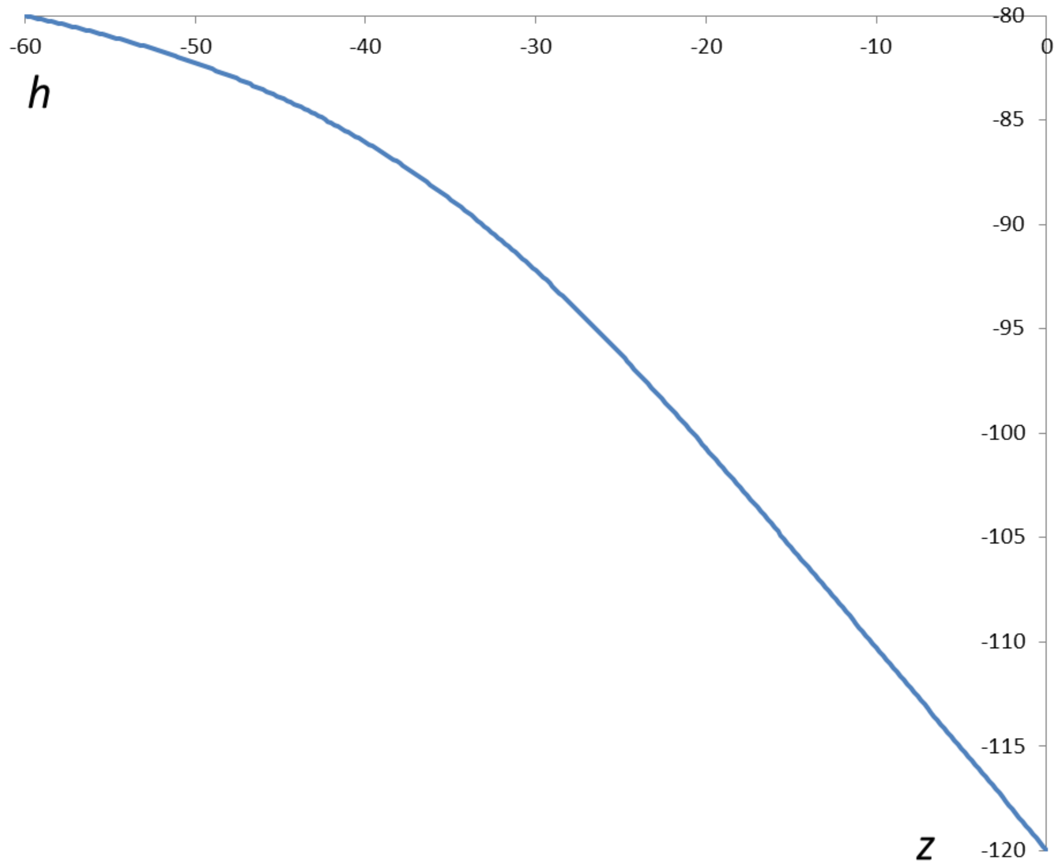
$$z = z_A - \int_{h(z_A)}^{h(z)} \frac{d\bar{h}}{1 + \frac{q}{K(\bar{h})}}$$

where z_B and $h(z_B)$ have been replaced by z and h(z). By selecting various values of h, corresponding values of z can be calculated.

The plot below is z vs. h, which shows clearly that the assumption of a linear variation of h with z is poor.



The same approach applied for the second part of the question can be used, the only difference is $h(z_A)$ = -60 and $h(z_B)$ = 0. In this case, the Darcy flux is 0.378 cm/day, which is closer to the value found above. In this case, the flow is upwards, i.e., against gravity, which is an additional "brake" on the Darcy flux. Thus, even though the range of K varies more, the approximation is less erroneous. This can been seen in the plot below, where it is shown that the assumption of h being linear in z is more reasonable.



Solution exercice 5_4 (extra challenge)

(a) Using the given form of the water capacity term we find that Richards' equation becomes:

$$-\frac{\alpha^*}{h-h_s}\frac{dK(h)}{dh}\frac{\partial h}{\partial t}=\frac{dK(h)}{dh}\frac{\partial h}{\partial z}\left(\frac{\partial h}{\partial z}-1\right)+K(h)\frac{\partial^2 h}{\partial z^2},$$

where $\frac{\partial K(h)}{\partial z}=\frac{dK(h)}{dh}\frac{\partial h}{\partial z}$. Use $\frac{d[\ln K(h)]}{dh}=\frac{1}{K(h)}\frac{dK(h)}{dh}$ to get:

$$-\frac{\alpha^*}{h-h_s}\frac{d[\ln K(h)]}{dh}\frac{\partial h}{\partial t}=\frac{d[\ln K(h)]}{dh}\frac{\partial h}{\partial z}\left(\frac{\partial h}{\partial z}-1\right)+\frac{\partial^2 h}{\partial z^2}.$$

Since h = hs + z/A(t), we have:

$$\frac{\partial h}{\partial z}=\frac{1}{A(t)},$$

$$\frac{\partial^2 h}{\partial z^2}=0$$

and

$$\frac{\partial h}{\partial t}=-\frac{z}{A(t)^2}\frac{dA(t)}{dt}=-\frac{h-h_s}{A(t)}\frac{dA(t)}{dt}.$$

The combination of the results given in this question gives:

$$\alpha^* \frac{dA(t)}{dt} = \frac{1}{A(t)} - 1 \quad (*),$$

which is the required differential equation.

(b) This result is easily shown by differentiating the given expression. The initial condition is satisfied, as can be seen by substitution.

(c) The potential is given by the expression H = h - z (z is positive downwards). Darcy's Law then gives:

$$q(z,t) = -K(h)\frac{\partial H}{\partial z} = -K(h)\frac{\partial}{\partial z}(h - z) = K(h)\frac{\partial}{\partial z}\left[h_s + \frac{z}{A(t)} - z\right] = K(h)\left[1 - \frac{1}{A(t)}\right].$$

Because the surface is always ponded, at z = 0 we have $K(h) = K[h(0,t)] = K(h_S) = K_S$, the saturated conductivity. Therefore:

$$q(0,t) = K_s\left[1 - \frac{1}{A(t)}\right].$$

(d) The differential equation for A(t) is, from above:

$$\alpha^* \frac{dA}{dt} = \frac{1}{A(t)} - 1$$

Use this to solve the integral for I(t):

$$I(t) = \int_0^t q(0,\bar{t})\, d\bar{t} = K_s \int_0^t \left[1 - \frac{1}{A(t)}\right] d\bar{t} = -\alpha^* K_s \int_0^t \frac{dA(\bar{t})}{d\bar{t}}\, d\bar{t} = -\alpha^* K_s A(t),$$

where A(0) = 0 has been used.

(e) Substitute the expression for I(t) in (d) in the governing equation for A(t):

$$K_s t = I(t) - \alpha^* K_s \ln\left[1 + \frac{I(t)}{\alpha^* K_s}\right],$$

in which we want to replace α^*. Use the expression given in the question to get:

$$K_s t = I(t) - \frac{S^2}{2K_s} \ln\left[1 + \frac{I(t)}{S^2/2K_s}\right].$$